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Cryptanalysis Lab 4

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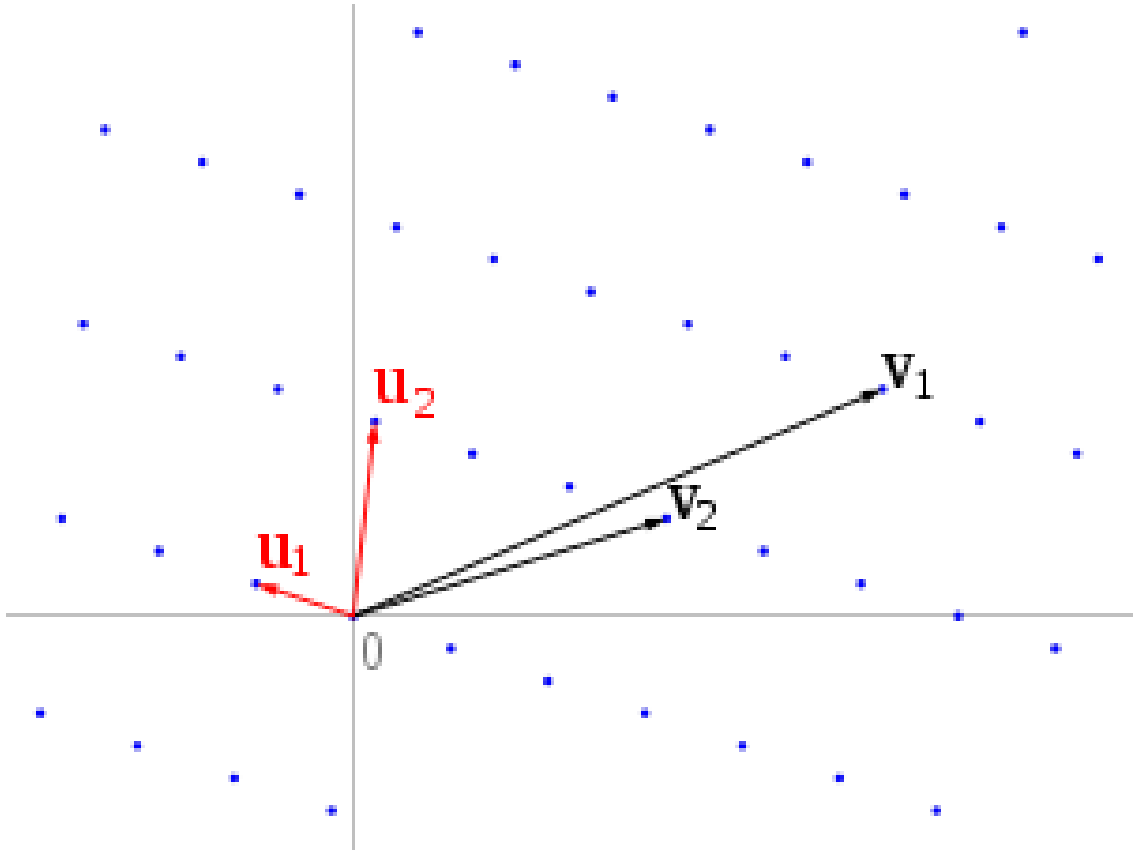
The LLL Algorithm

Given a set of basis vectors $\mathcal{S} = \mathbf{x}_1, \ldots, \mathbf{x}_n$ with integer entries, we can form a lattice $\mathcal{L}$ by taking all integer linear combinations of vectors in $\mathcal{S}$. The picture below shows a lattice generated by vectors in $\mathbb{Z}^2$. The same lattice can be generated by $\mathbf{u}_1, \mathbf{u}_2$, or by $\mathbf{v}_1, \mathbf{v}_2$.



The LLL algorithm takes a collection of 'bad' basis vectors for

lattice, such as $\mathbf{v}_1, \mathbf{v}_2$, and tries to generate a set of 'good' basis vectors for the lattice, such as $\mathbf{u}_1, \mathbf{u}_2$, which are much shorter, and close to being perpendicular.

In the next two questions, you will use the LLL algorithm to solve problems related to cryptography.

A Knapsack-Based Hash Function

We can try to construct a hash function based on the hardness of solving the Knapsack problem.

Let $a_1, \ldots, a_n$ be positive integers. Given a positive integer s, we might ask whether there exist $x_1, \ldots, x_n \in \{0, 1\}$ such that $\sum_{i=1}^{n} a_i x_i = s$. This is a special case of the knapsack problem, and it is NP-complete, which suggests that the function $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} a_i x_i$ might be difficult to invert, and have some of the properties of a good hash function.

Concretely, we create a hash function by choosing random values for the $a_1, \ldots, a_n$. We hash values $(x_1, \ldots, x_n) \in \{0, 1\}^n$ to $\{0, 1\}^k$ by computing $s = \sum_{i=1}^{n} a_i x_i$, and then taking the binary digits of s as

output.

Click on the green letter before each question to get a full solution. Click on the green square to go back to the questions.

Exercise 1.

(a) Implement a function 'Parameters' which takes integers n, μ as input, and generates n random μ-bit integers $a_1, \ldots, a_n$ for use in a knapsack-based hash function.

(b) Create a function 'KnapsackHash' which implements the knapsack-based hash function described above. Your function should take the output of part a) and a value to hash, and produce a hash value.

(c) Consider the lattice generated by the rows of the following matrix, for some large value K.

$$\begin{pmatrix} Ka_1 & 1 & 0 & \cdots & 0 \\ Ka_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Ka_n & 0 & 0 & \cdots & 1 \end{pmatrix}$$

How might finding a short vector in this lattice help to find a collision in the knapsack hash function?

Hint: What happens if we find a short vector with first component zero, and the other components 1 or -1?

(d) The LLL algorithm can be applied to a matrix M by writing M.LLL(). On input a square matrix of row vectors, the LLL algorithm produces a new matrix, where the first row is a short vector in the lattice. Write a program which uses the LLL algorithm to break a knapsack hash function with `params = Parameters(n,mu)` for $(n, mu) = (10, 10), (20, 20), (40, 40)$. What are the largest values of (n, mu) for which your program finds a collision?

Finding Polynomials with Small Coefficients from Approximate Roots

Taken from Algorithmic Cryptanalysis, Chapter 13, Exercise 1:
Consider the floating point number:

$$x = 8.443116105837945503931385 17.$$

Show that x is a close approximation of a real root of a polynomial of degree 3, with integer coefficients bounded by 20 (in absolute value).

Click on the green letter before each question to get a full solution. Click on the green square to go back to the questions.

EXERCISE 2.

(a) With the collision matrix from the previous question in mind, design a matrix containing the powers of x, where a short vector in the lattice is likely to produce a polynomial of degree 3, with x as a root.

(b) Apply the LLL algorithm to your matrix to find the polynomial.

Elliptic Curve Factorisation Algorithm

Click on the green letter in front of each sub-question (e.g. (a)) to see a solution. Click on the green square at the end of the solution to go back to the questions.

EXERCISE 3. In this exercise, you will use Sage to explore how integers are factored using elliptic curves.

(a) To create an elliptic curve Ep defined by $y^2 = x^3 + ax + b$ over $\mathbb{F}_p$, use `E = EllipticCurve(GF(p),[a,b])`. Create an elliptic curve Ep defined by $y^2 = x^3 + x + 4$, over the finite field of size 11.

(b) To create a curve EN defined by $y^2 = x^3 + ax + b$ over $\mathbb{Z}_N$, use `E = EllipticCurve(Integers(N),[a,b])`. Create a curve EN defined by $y^2 = x^3 + x + 4$, over the ring of integers modulo 438713.

(c) Type $PN = EN(100584, 115601)$ to create the corresponding point on EN. Similarly, type $Pp = Ep(100584, 115601)$ to create the same point, reduced modulo 11, on Ep. Type `Pp` to view the point modulo 11. The point should be expressed as $(x : y : 1)$. The point at infinity is $(0 : 1 : 0)$.

(d) Type `Ep.cardinality()` to find out the number of elliptic curve points modulo 11. What is the number of points? Type `a*Pp` to compute multiples of the point Pp. What is the order of Pp in the elliptic curve group Ep?

(e) Set $QN = 8 * PN$ and use SAGE to compute QN. Now, try to compute $9 * PN = QN + PN$. What happens? Compute the difference between the x coordinates of PN and QN, and compute the greatest common divisor of this with N. Look at the formulae

for adding elliptic curve points. Does this explain the error?

(f) Set $N = 20077$. Consider the curve E defined by the equation $y^2 = x^3 + x + 5$. Assume that N has a prime factor p with $| E(\mathbb{Z}_p) |$ being 5-powersmooth. Given that $P = (427, 466)$ is a point on $E(\mathbb{Z}_N)$, factor N.

Solutions to Exercises

Exercise 1(a) The following code implements the 'Parameters' function.

```
def Parameters(n,mu):
    A = list();
    for i in range(0,n):
        A.append(randint(0,2**mu-1))
    return [vector(A),n,mu]
```

□

Exercise 1(b) The following code implements the knapsack hash function.

```
def KnapsackHash(params,x):
    A = params[0]
    n = params[1]
    mu = params[2]
    k = ceil(log(n,2))+mu
    if n != len(x):
        return "The input vectors are not the same length!"
    z = 0;
    for i in range(0,len(x)):
        z = z + A[i]*x[i];
    z = z.bits()
    while len(z) < k:
        z.append(0)
    return z
```

□

Exercise 1(c) Following the hint, since K is large, short vectors in the lattice are likely to have first component equal to zero. Otherwise, the first component would be a large number, as a multiple of K. This means that a short vector is likely to involve finding a linear combination of the a_i which is equal to zero. The other components of the vector tell us the coefficients in this linear combination. If the other components are all 1 or -1, then we can rearrange the linear combination to find two binary inputs which hash to the same output value. □

Exercise 1(d) The following code implements a collision finder.

```
def BreakKnapsackHash(params,K):
#Choose large positive integer K
    A = params[0]
    n = len(A)
    B = matrix(A).transpose()
    C = matrix.identity(n)
    M = block_matrix([[K*B,C]])
    L = matrix(list(M.LLL()))
    L = L[0]
    if L[0] != 0:
        return 'fail'
    L = list(L)          #(continued on next page)
    L.remove(0)
    for entry in L:
```

```
        if abs(entry) > 1:
            return 'fail'
    X1 = matrix([[abs(L[i]>0) for i in range(0,len(L))]])
    X2 = matrix([[abs(L[i]<0) for i in range(0,len(L))]])
    return matrix(list(block_matrix([[X1],[X2]])))
```

□

Exercise 2(a)

$$\begin{pmatrix} \lfloor K \rceil & 1 & 0 & 0 & 0 \\ \lfloor Kx \rceil & 0 & 1 & 0 & 0 \\ \lfloor Kx^2 \rceil & 0 & 0 & 1 & 0 \\ \lfloor Kx^3 \rceil & 0 & 0 & 0 & 1 \end{pmatrix}$$

Apply the LLL algorithm to the lattice. The first element in our reduced basis has the form $(\epsilon, a_0, a_1, a_2, a_3)$, where $\epsilon = a_0\lfloor K \rceil + a_1\lfloor Kx \rceil + a_2\lfloor Kx^2 \rceil + a_3\lfloor Kx^3 \rceil$ and ϵ is quite small. Dividing by K, this suggests that x is a close approximation to a root of the polynomial with coefficients a_i. □

Exercise 2(b) Section 13.1.2.2 of Algorithmic Cryptanalysis suggests using $K \geq (\max |a_i|)^{2d}$ where d is the degree of the polynomial. So in our example we can take $K = (20)^6 = 64 \times 10^6$. This gives us a matrix:

$$\begin{pmatrix} 64000000 & 1 & 0 & 0 & 0 \\ 540359431 & 0 & 1 & 0 & 0 \\ 4562317413 & 0 & 0 & 1 & 0 \\ 38520175629 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Apply the LLL algorithm. For your own sanity, use SAGE rather than trying to do LLL by hand. Create a matrix A as above, do A.LLL() and look at the first row. This is $(-3, -10, -11, -7, 1)$, corresponding to $x^3 - 7x^2 - 11x - 10$. Check for yourself that $x^3 - 7x^2 - 11x - 10$ is extremely close to 0. □

Exercise 3(a) Use `Ep = EllipticCurve(GF(11),[1,4])` to produce the correct elliptic curve. □

Exercise 3(b) Use `EN = EllipticCurve(Integers(438713),[1,4])` to produce the correct curve. □

Exercise 3(d) There are 9 points on the elliptic curve Ep defined modulo 11. The order of a group element divides the order of the group, 9, and Pp is not the point at infinity, so we only have to check whether $3 * Pp$ is equal to the point at infinity or not. However, $3 * Pp = (3 : 1 : 1)$, so 3 is not the order of Pp. Therefore, the order of Pp is 9. □

Exercise 3(e) You can use `a = randint(0,1009)` to get a. □